LAKSHYA

JEE 2024

MATHEMATICS

APPLICATION OF DERIVATIVES

Lecture -14

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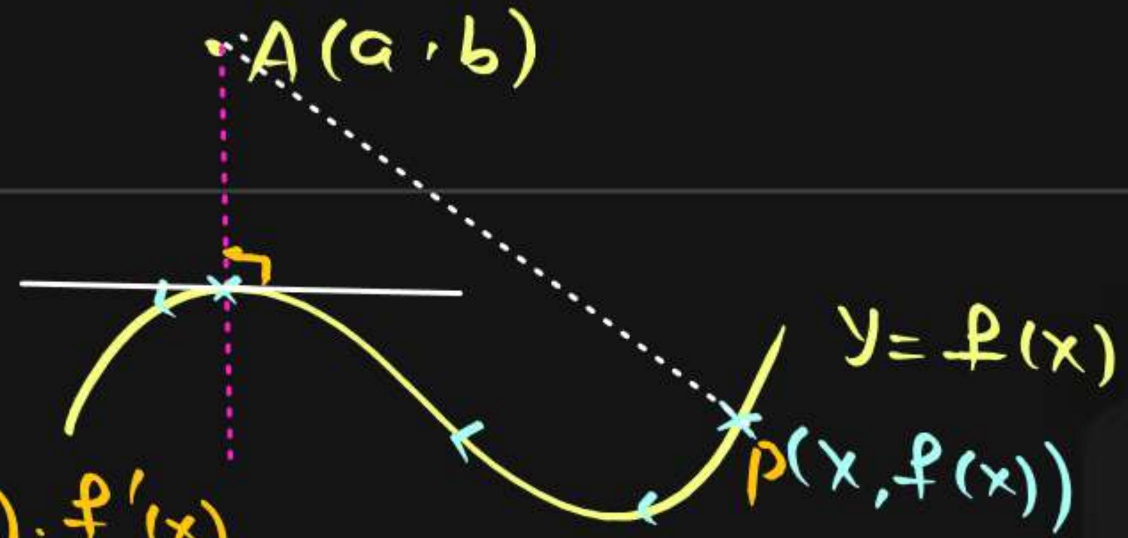
RECAP OF PREVIOUS LECTURE

Topic An Important Point

Topic Curve Tracing

An Important Concept

Given a fixed point A(a, b) and a moving point P(x, f(x)) on the curve y = f(x). then AP will be maximum or minimum if it is normal to the curve at P.



$$D = AP^2 = (x-a)^2 + (f(x)-b)^2$$

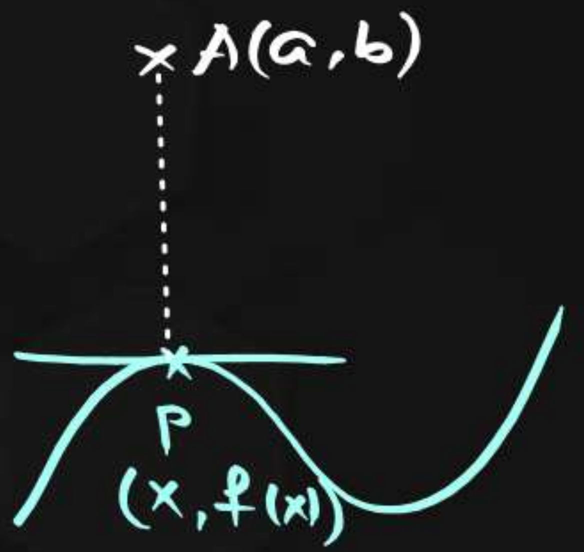
$$\frac{dD}{dx} = 2(x-a) + 2(f(x)-b).f'(x)$$

for MAX|MIN $\frac{dD}{dx} = 0 \Rightarrow \cancel{2}(x-a) + \cancel{2}(f(x)-b)(f'(x)) = 0$

$$f'(x) = \frac{-(x-a)}{f(x)-b}.$$

$\Rightarrow$ AP is max or min if $f'(x) = -\left(\frac{x-a}{f(x)-b}\right)$

i.e if $f'(x) \cdot \frac{f(x)-b}{x-a} = -1$

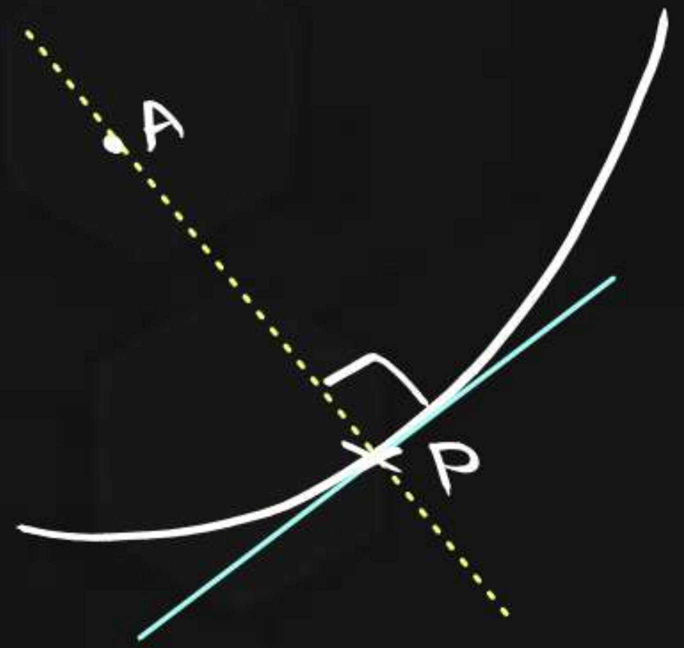


Slope of Tangent at P . Slope of AP = -1

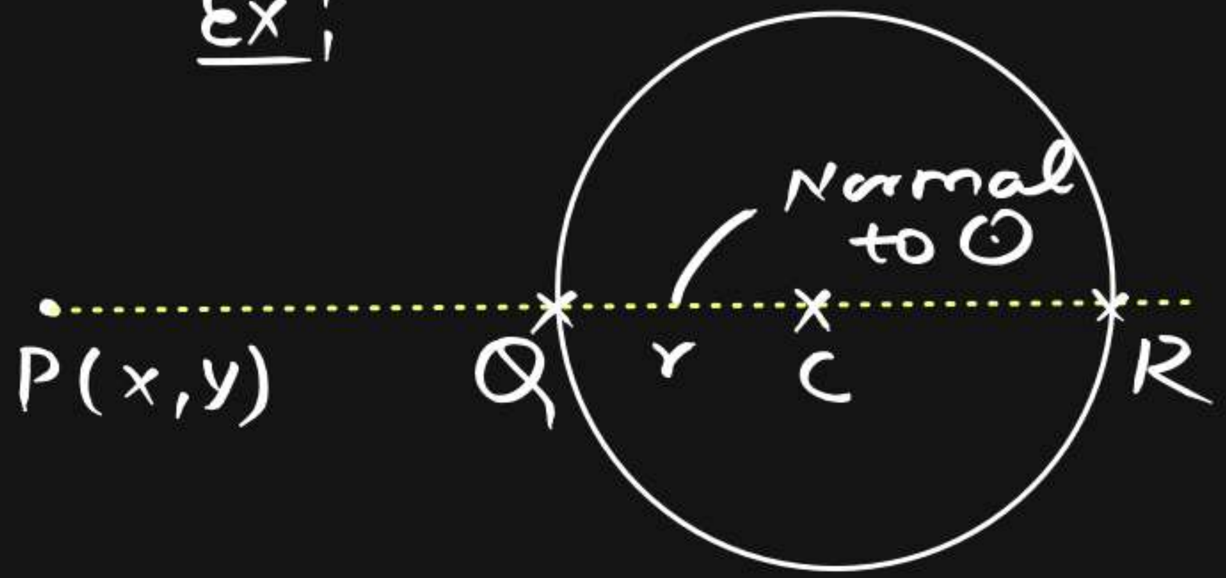
$m_T \cdot m_{AP} = -1$

$\Downarrow$

A is Normal to y = f(x) at P for AP to be max | min.

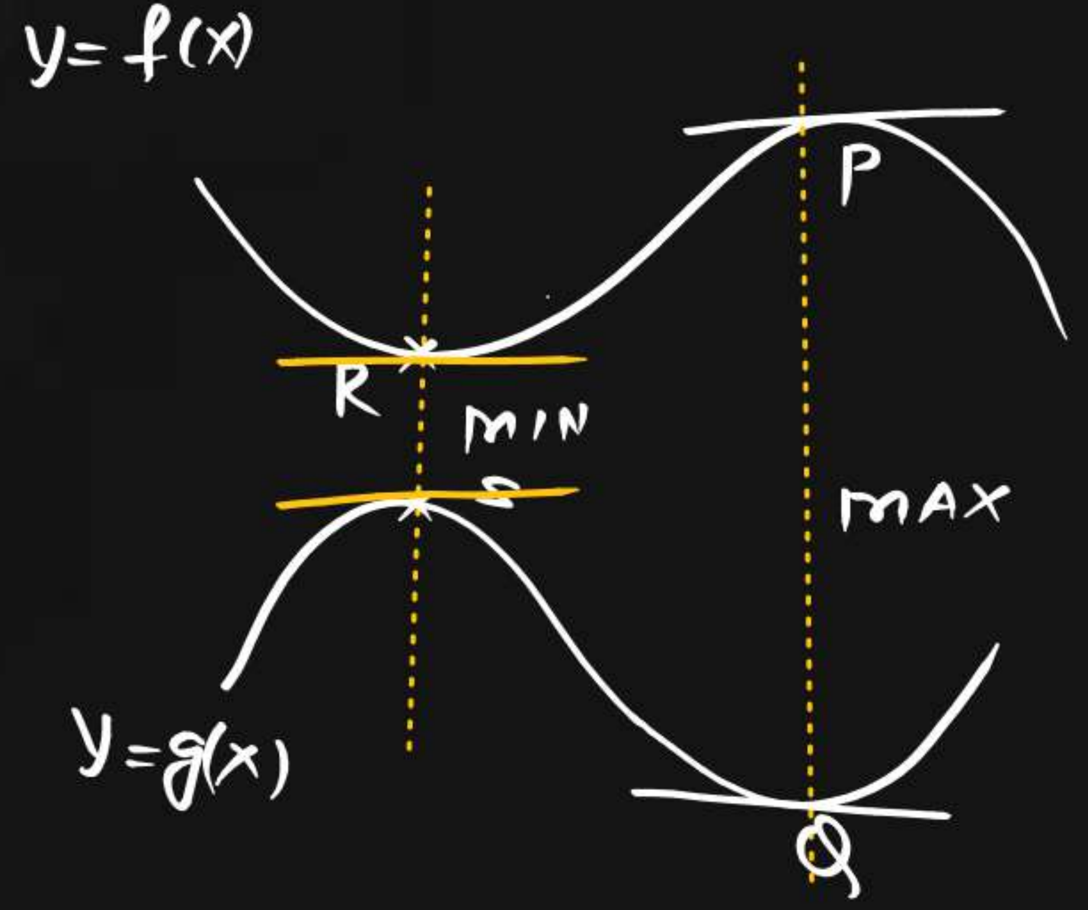


Ex :



MIN distance of P from circle = PQ

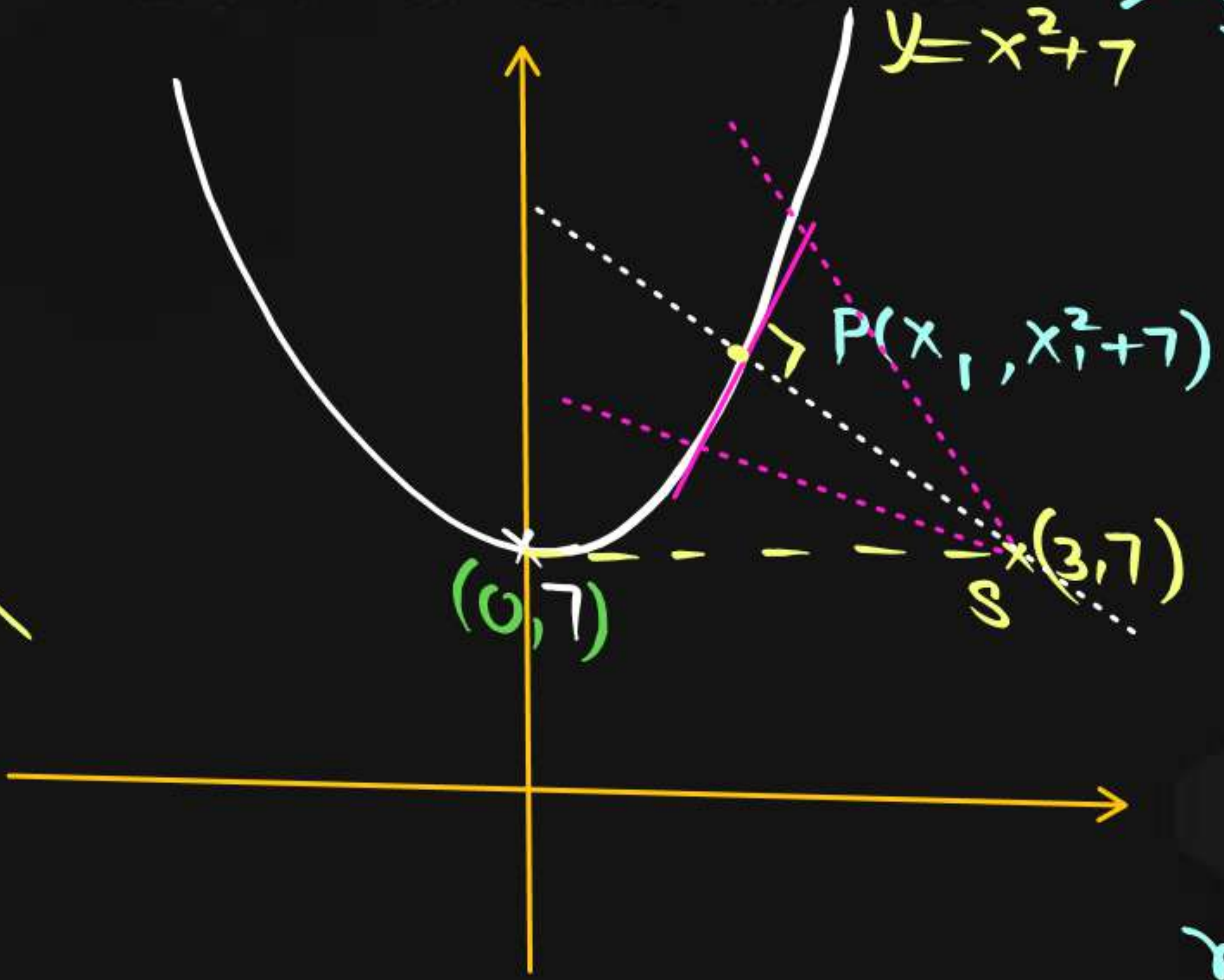
MAX distance of P from ⊙ = PR.



Min/MAX Distance
b/w two curves
occur along
Common Normal

Question

#Q. A helicopter of enemy is flying along the curve given by $y = x^2 + 7$. A soldier placed at (3, 7) wants to shoot down the helicopter when it is nearest to him. Find the nearest distance.



$\frac{dy}{dx} = 2x$

Distance will be min If SP is normal to $y = x^2+7$ at P.

$$m_{SP} = \frac{x_1^2+7-7}{x_1-3} = \frac{x_1^2}{x_1-3}$$

$$m_T = \left.\frac{dy}{dx}\right|_{x=x_1} = \left.2x\right|_{x=x_1} = 2x_1$$

$m_{ST} \cdot m_T = -1$

$\frac{x_1^2}{x_1 - 3} \cdot 2x_1 = -1$

$2x_1^3 = -x_1 + 3$

$2x_1^3 + x_1 - 3 = 0$

$2x_1^2(x_1 - 1) + 2x_1(x_1 - 1) + 3(x_1 - 1) = 0$

$(2x_1^2 + 2x_1 + 3)(x_1 - 1) = 0$

$x_1 = 1$, $\underbrace{2x_1^2 + 2x_1 + 3 = 0}_{(N.P)}$ $D < 0$. $\Rightarrow P(1, 8)$

MIN Distance $= SP = \sqrt{(3-1)^2 + (7-8)^2}$

$= \sqrt{5}$.

$x = 1.1 \quad y_1 = x_1^2 + 7 = 8.21$

$\sqrt{(1.9)^2 + (1.21)^2}$

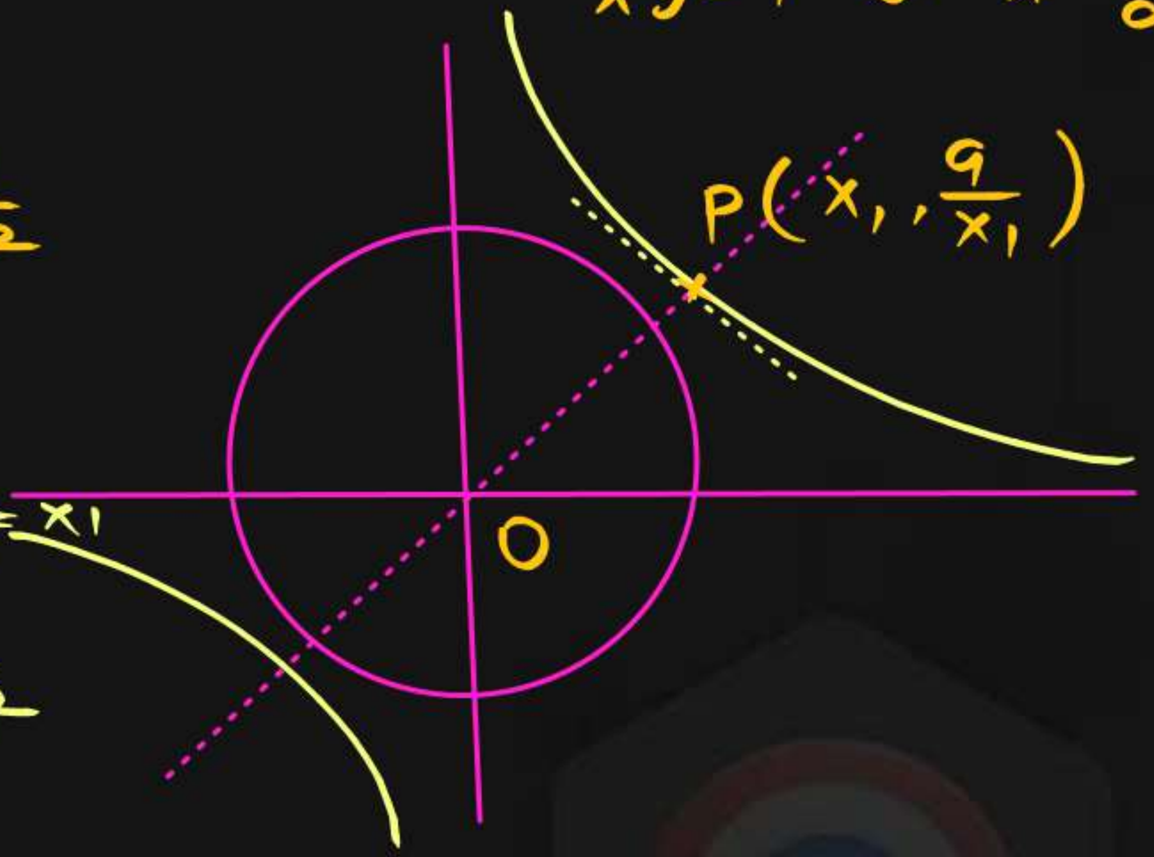
$\sqrt{3.61 + 1.44}$

$\sqrt{5.05}$.

Question

#Q. Find the minimum distance between $x^2 + y^2 = 2$ and $xy = 9$.

$xy = 9 \rightarrow y = \frac{9}{x} \rightarrow \frac{dy}{dx} = \frac{-9}{x^2}$



$$m_{OP} = \frac{\frac{9}{x_1} - 0}{x_1 - 0} = \frac{9}{x_1^2}$$

Slope of Tangent at $P = m_T = \left.\frac{dy}{dx}\right|_{x=x_1}$

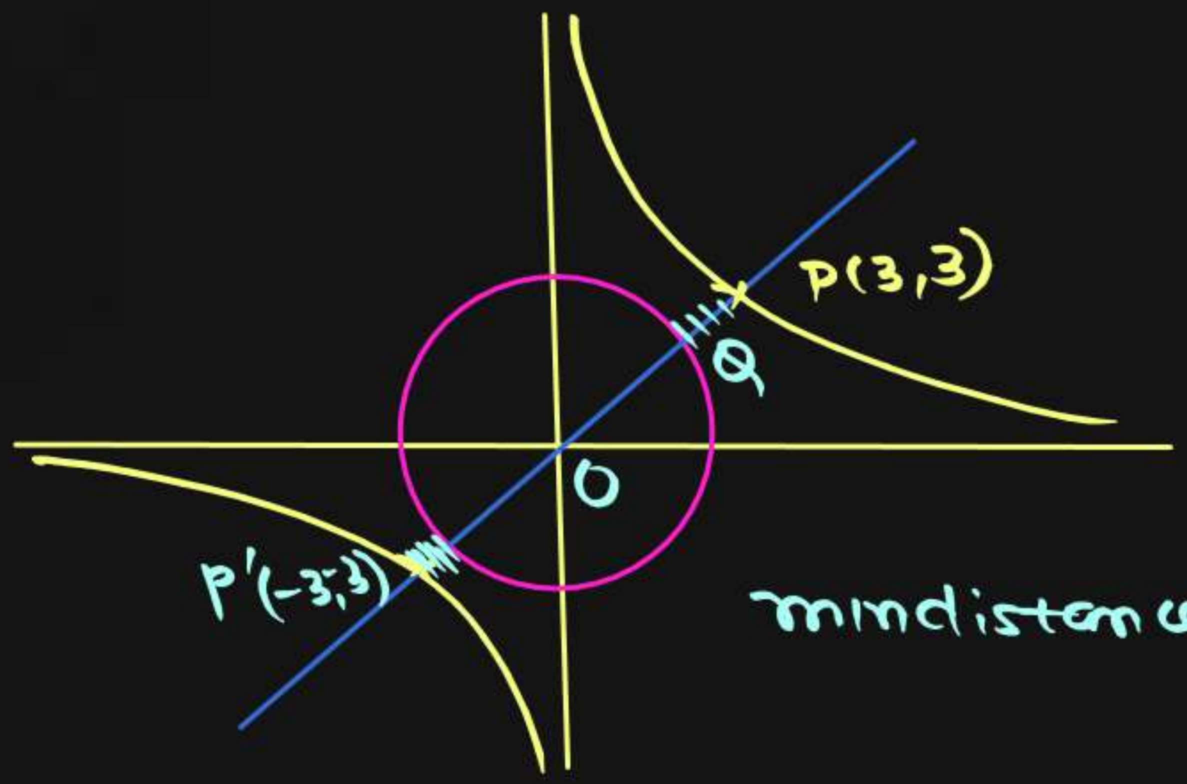
$$= \left.-\frac{9}{x^2}\right|_{x=x_1} = -\frac{9}{x_1^2}$$

$$m_{OP} \cdot m_T = -1$$

$$\frac{9}{x_1^2} \cdot \frac{-9}{x_1^2} = -1$$

$$x_1^4 = 81 \Rightarrow x_1^2 = 9 \Rightarrow x_1 = 3, -3$$

possible coord of $P = (3,3)$ or $(-3,-3)$



min distance $= PQ = OP - r = OP' - r$

$= \sqrt{3^2+3^2} - \sqrt{2}$

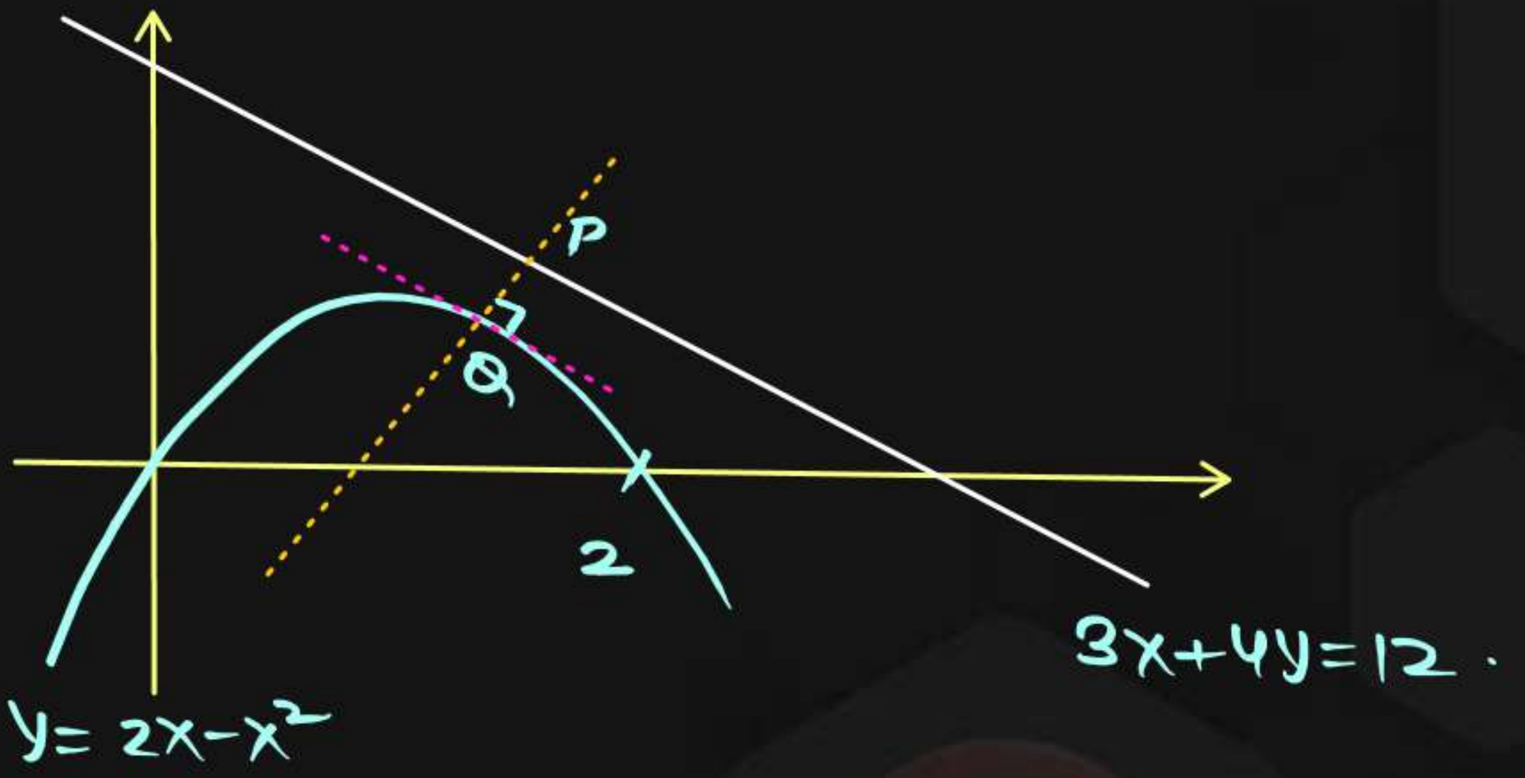
$= 3\sqrt{2} - \sqrt{2} = \underline{2\sqrt{2}}$

Question

#Q. Find the coordinates of a point on $y = 2x - x^2$, which is at least distance from the line, $4x + 3y = 12$.

$$L: \frac{x}{4} + \frac{y}{3} = 1$$

TAH !



Question

#Q. $\text{Min}\left[(x_1-x_2)^2+\left(\underbrace{7+\sqrt{4-(x_1+4)^2}}_{y_1}-\underbrace{\sqrt{4x_2}}_{y_2}\right)^2\right], \forall\, x_1, x_2 \in R$ is

A $5-\sqrt{2}-2$

B 5

C $5\sqrt{3}-2$

D 10

$C_1 : y_1 = f(x_1)$, $C_2 : y_2 = f(x_2)$

$\downarrow$ P (x_1, y_1) $\quad$ $\downarrow$ Q (x_2, y_2)

$$PQ^2 = (x_1-x_2)^2+(y_1-y_2)^2$$

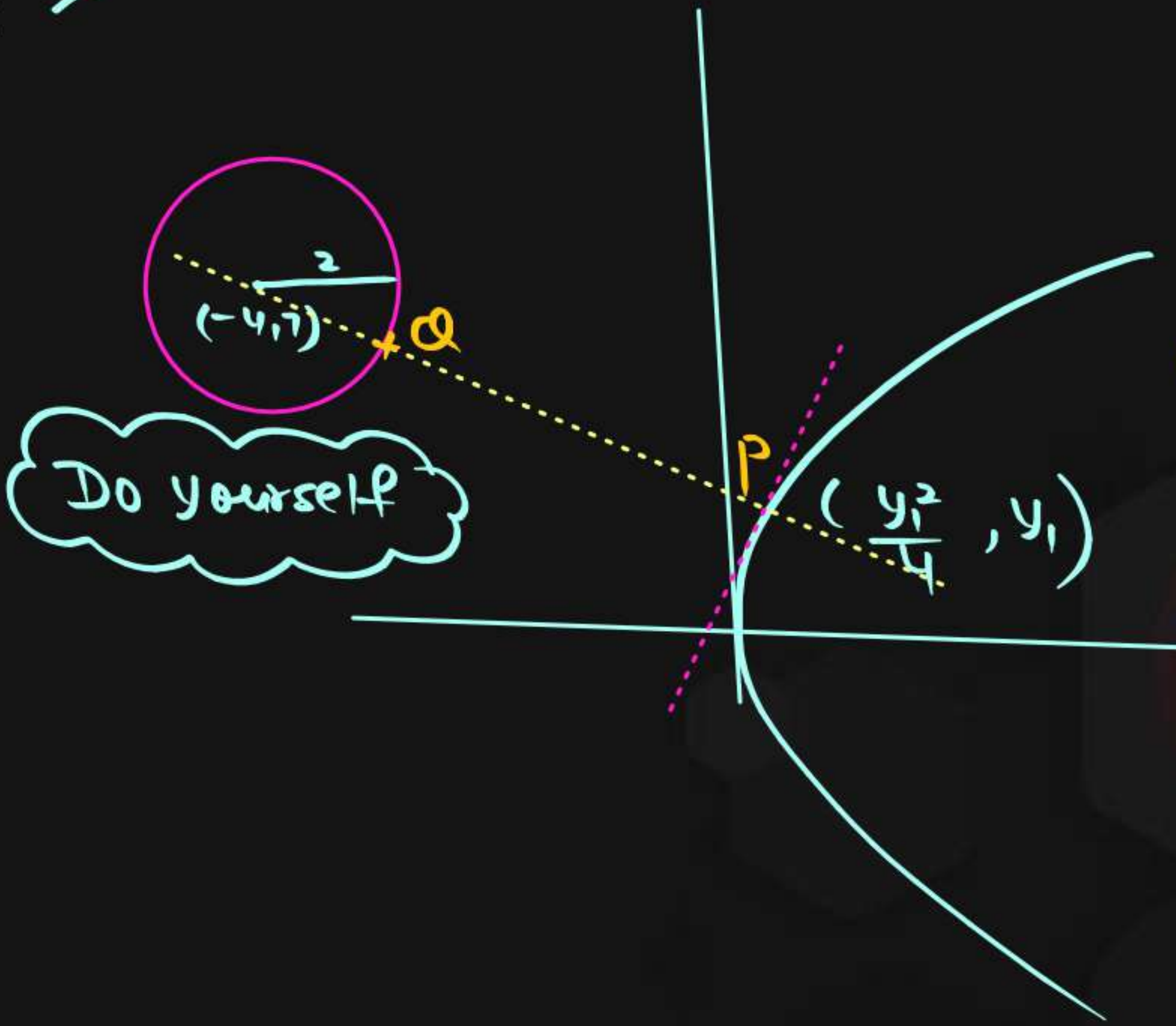
$$y_1 = 7+\sqrt{4-(x_1+4)^2} \Rightarrow (y_1-7)^2 = 4-(x_1^2+4)^2 \Rightarrow (x_1+4)^2+(y_1-7)^2 = 4$$

$$y_2 = \sqrt{4x_2} \Rightarrow y_2^2 = 4x_2$$

$C_1 : (x+4)^2+(y-7)^2=4$

$C_2 : y^2=4x$

Now find(min distance)^2



$y^2=4x$.

$P(x_1, y_1)$

$y_1^2=4x_1$

$x_1=y_1^2/4$

$\Downarrow$

$P(y_1^2/4, y_1)$

Topic : Curve Tracing

The following checklist is intended as a guide to sketching a curve $y = f(x)$.

A. **Domain** : find domain it gives region in which curve present.

B. **Intercepts** : find points where curve intersects x axis & y axis

for $x_{\text{intercept}}$ put $y=0$ & solve for x

$y_{\text{intercept}}$ put $x=0$ & find y.

C. **Symmetry** : find if fn is odd/even.

D. **Asymptotes** :

E. **Intervals of Increase/Decrease**

F. **Local Maximum and Minimum Value**

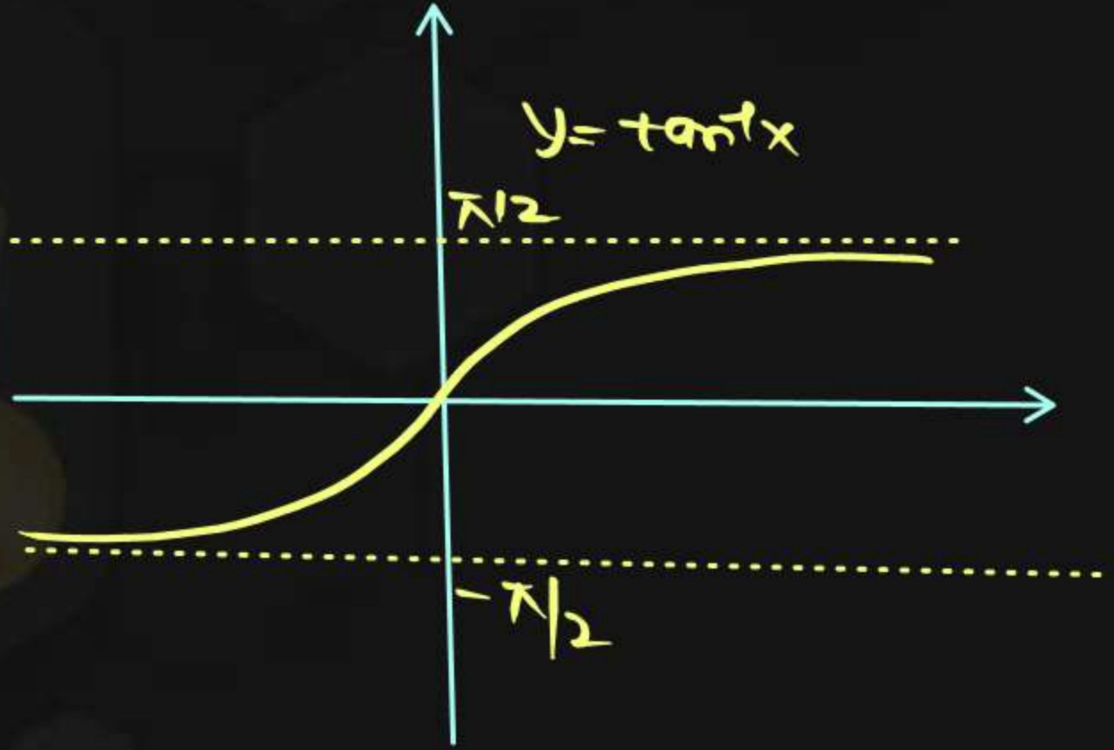
G. **Concavity and Points of Inflection**

H. **Sketch the Curve**

Asymptotes: for the fn $y=f(x)$

(1) Horizontal Asymptote: If $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to -\infty} f(x) = b$

then $y=b$ is a horizontal asymptote.



Ex: $y=e^x \rightsquigarrow \lim_{x\to-\infty} e^x = e^{-\infty} = 0 \Rightarrow y=0$ is a horizontal asymptote.

$y=\tan^{-1}x$ $\lim_{x\to\infty} \tan^{-1}x = \pi/2 \Rightarrow y=\pi/2$ is horizontal Asymptote.

$\lim_{x\to-\infty} \tan^{-1}x = -\pi/2 \Rightarrow y=-\pi/2$ is another horizontal Asymp

(2) vertical Asymptote; If for fn $y=f(x)$

$\lim\limits_{x\to a^-} f(x)$ or $\lim\limits_{x\to a^+} f(x)=\infty$ or $-\infty \Rightarrow x=a$ is a vertical Asympt.

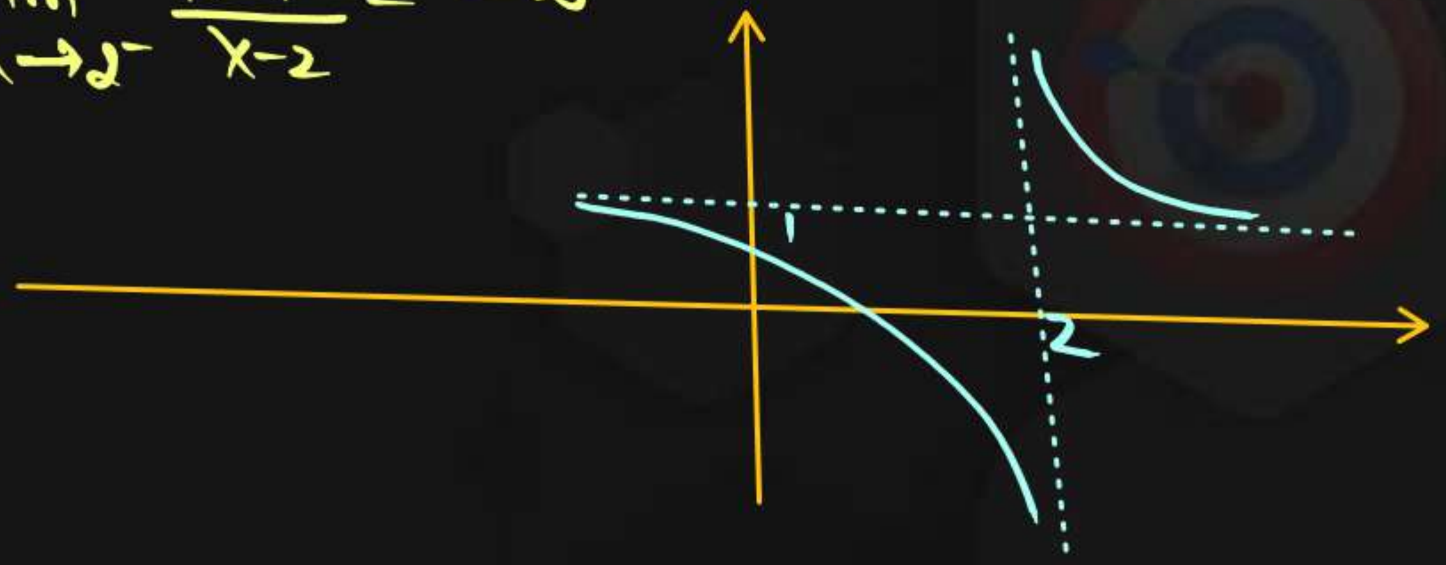
Ex: $y=\dfrac{x-1}{x-2}$

$\lim\limits_{x\to 2^+}\dfrac{x-1}{x-2}=\infty$

$\lim\limits_{x\to 2^-}\dfrac{x-1}{x-2}=-\infty$

$\Big\}\ x=2$ is a vertical Asymptote.

$\lim\limits_{x\to\infty}\dfrac{x-1}{x-2}=1 \Rightarrow y=1$ is horizontal Asymptote.



Question

#Q. Find the intervals of monotonocity for the following functions & represent your solution set on the number line.

(a) $f(x) = 2 \cdot e^{x^2-4x}$ (b) $f(x) = e^x/x$

(c) $f(x) = x^2e^{-x}$ (d) $f(x) = 2x^2 - \ln|x|$

Also plot the graphs in each case & state their range.

(1) $y = f(x) = 2e^{x^2-4x}$

(1) Domain $= R$

(2) Neither odd / even

(3) $x=0 \quad y=2$

$y=0$ N.P $\rightarrow$ does not intersect X Axis.

(4) No vertical or horizontal Asympt

(5) $\frac{dy}{dx} = 2\cdot(2x-4)e^{x^2-4x}$

$= 4(x-2)e^{x^2-4x}$

MIN

$-$ $+$

2

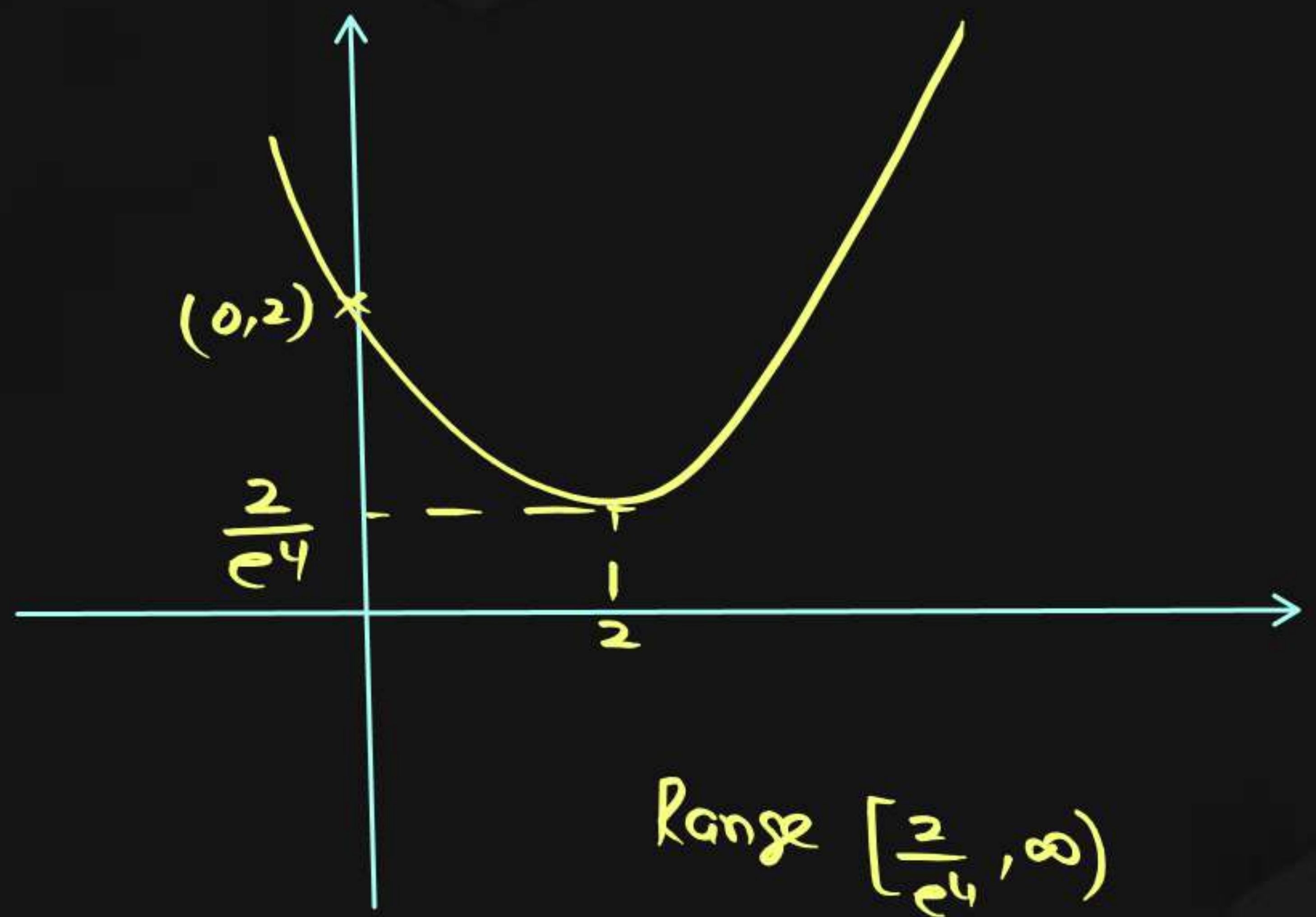
(6) $\frac{d^2y}{dx^2} = 4e^{x^2-4x} + 4(x-2)\cdot(2x-4)e^{x^2-4x}$

$\frac{d^2y}{dx^2} = 4e^{x^2-4x}(1+2(x-2)^2)$

$= +ve$ (concave up on R)

$y = 2e^{x^2-4x}$

$y(2) = \frac{2}{e^4}$



Range $\left[\frac{2}{e^4}, \infty\right)$

Many one.

(2) $y=\frac{e^x}{x}$ (1) Domain: $\mathbb{R}-\{0\}$.

(2) $x=0$ N.P $\rightsquigarrow$ curve does not intersect y axis

$y=0$ N.P $\rightsquigarrow$ curve does not intersect x axis

(3) as $x\to 0^+$ $\lim_{x\to 0^+}\frac{e^x}{x}=+\infty$

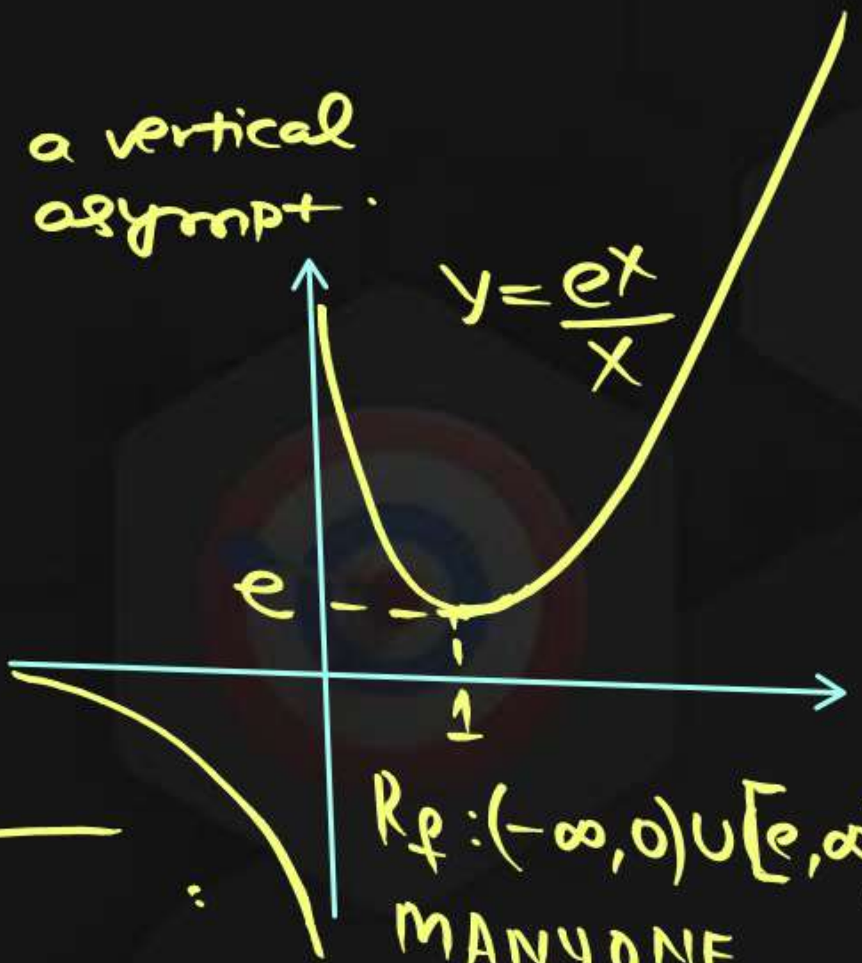
$\Rightarrow x=0$ is a vertical asympt.

$x\to 0^-$ $\lim_{x\to 0^-}\frac{e^x}{x}=-\infty$.

$\lim_{x\to -\infty}\frac{e^x}{x}=e^{-\infty}\cdot\frac{1}{-\infty}$
$=0\cdot 0$
$=0$

$\lim_{x\to\infty}\frac{e^x}{x}\left(\frac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{e^x}{1}=\infty$

(4) $\frac{dy}{dx}=\frac{x\cdot e^x-e^x}{x^2}=\frac{(x-1)e^x}{x^2}$



$R_f:(-\infty,0)\cup[e,\infty)$
MANYONE
INTO.

(3) $f(x) = x^2 \cdot e^{-x}$

(1) Domain $= R$

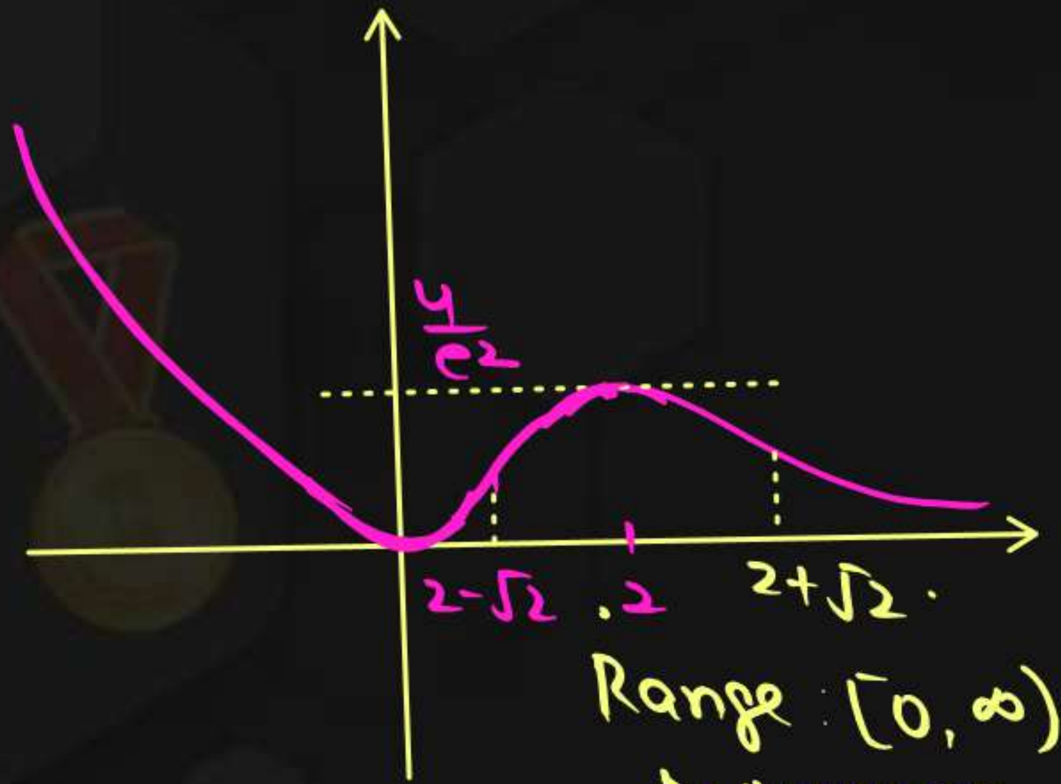
(2) $y = 0 \Rightarrow x = 0$
$x = 0 \Rightarrow y = 0$ $\Big\}$ passes $(0,0)$

(3) $\lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0 \Rightarrow y = 0$ is horizontal asymptote.

$\Downarrow$

i.e x Axis is horizontal Asymp

$\lim_{x \to -\infty} \frac{x^2}{e^x} = \infty \cdot \frac{1}{e^{-\infty}} = \infty \cdot \infty = \infty$



Range : $[0, \infty)$
MANYONE

(4) $\frac{dy}{dx} = 2xe^{-x} - x^2 e^{-x} = e^{-x}(2x - x^2)$

$= e^{-x} \cdot x(2-x) = e^{-x}(2x - x^2)$

$f(2) = \frac{4}{e^2}$

$f(0) = 0$

$\frac{d^2y}{dx^2} = e^{-x}(2-2x) - e^{-x}(2x - x^2)$

$$\frac{d^2y}{dx^2} = e^{-x}(2-2x-2x+x^2)$$

$$= e^{-x}(2-4x+x^2) = 0$$

$$x^2 - 4x + 2 = 0$$

$$x = \frac{4 \pm \sqrt{8}}{2}$$

$$x = 2 \pm \sqrt{2}$$

Question

#Q. Plot the following curves :

(a) $y = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + 6$

(b) $y = \frac{x}{\ln x}$

(c) $y = x \ln x$

(d) $y = \frac{\ln x}{x}$

(e) $y = \frac{x+1}{(x-1)(x-7)}$

TAH 3

① $x = 0, y = 6$

② $\frac{dy}{dx} = x^2 - 3x + 2 = (x-1)(x-2)$

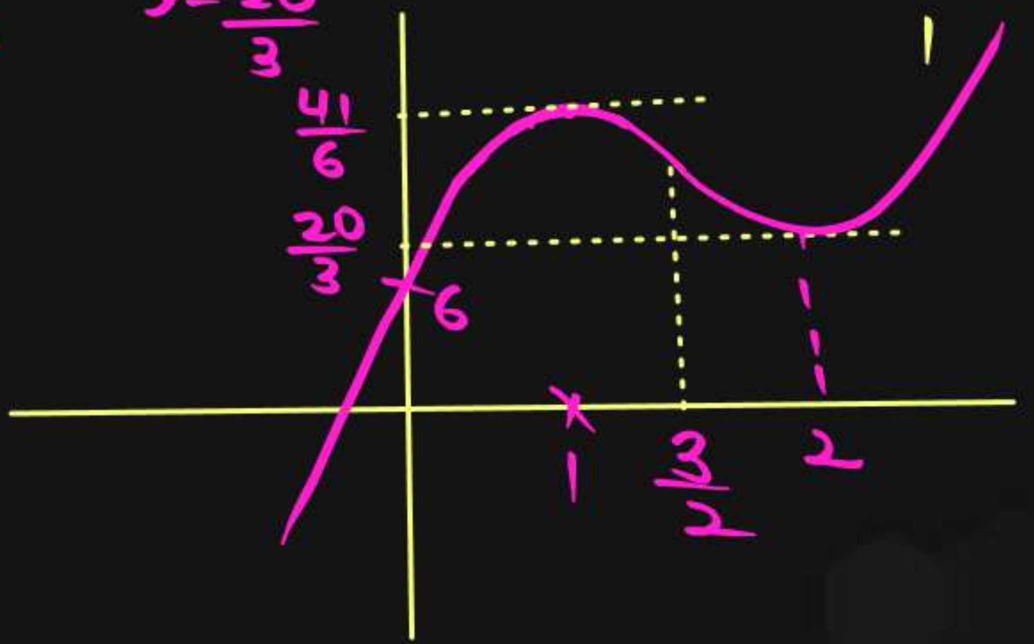
MAX MIN

$+ \quad - \quad +$

$1 \quad 2$

$x = 1$
$y = \frac{41}{6}$

$x = 2$
$y = \frac{20}{3}$



$\frac{d^2y}{dx^2} = 2x - 3 = 0$

$x = 3/2$

$\Downarrow$

$x = 3/2$ inflexion.

Question

#Q. Sketch a possible graph of a differentiable function f that satisfies the following conditions :

(i) $f'(x) > 0$ on $(-\infty, 1) \Rightarrow \uparrow (-\infty, 1)$

$f'(x) < 0$ on $(1, \infty) \Rightarrow \downarrow (1, \infty)$

$x = 1$ ↓ MAX

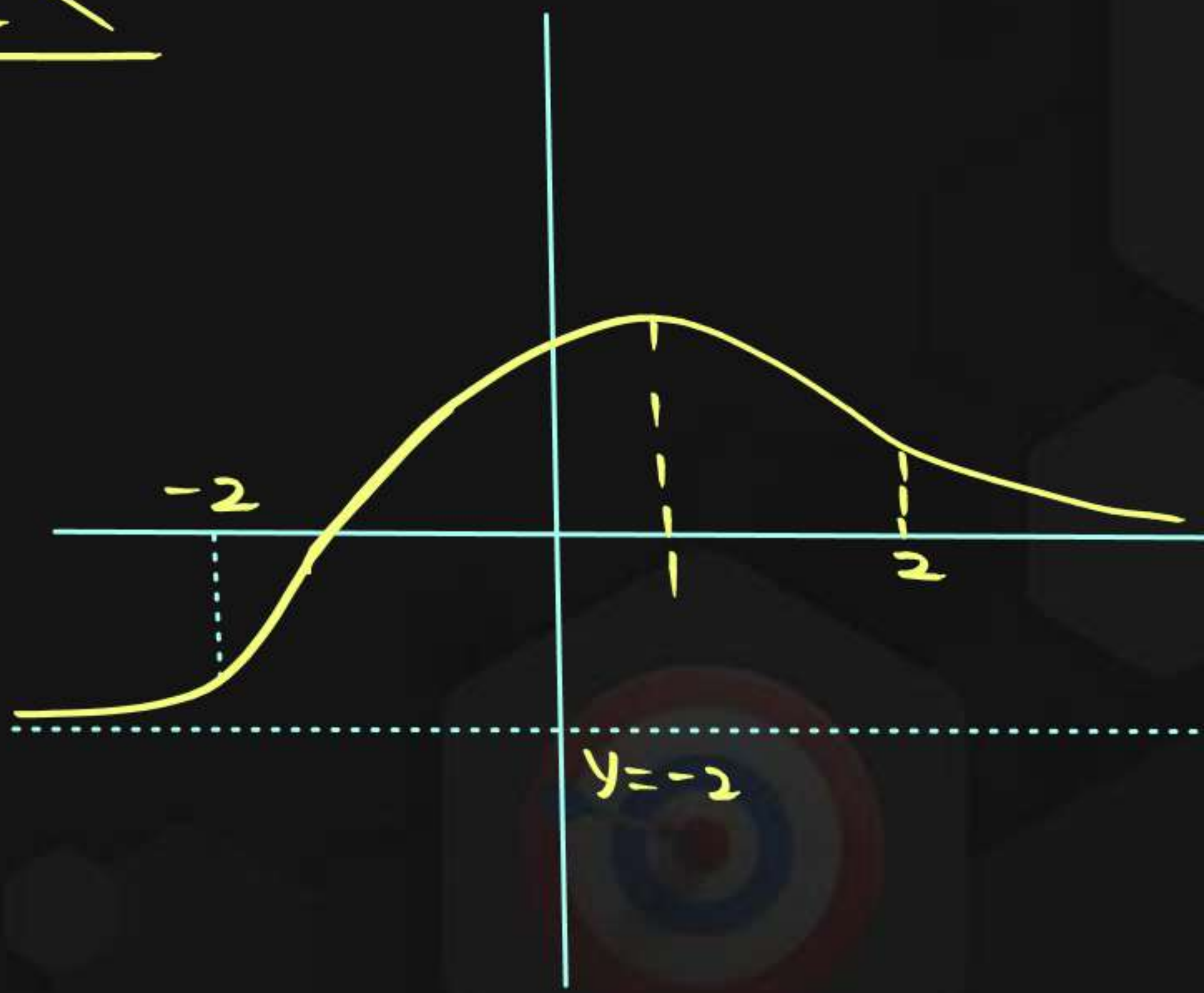
(ii) $f''(x) > 0$ on $(-\infty, -2) \rightarrow$ concave up in $(-\infty, -2) \cup (2, \infty)$.

and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$ — concave down in $(-2, 2)$

(iii) $\lim\limits_{x \to -\infty} f(x) = -2, \quad \lim\limits_{x \to \infty} f(x) = 0$

$y = -2$ is a horizontal Asympt

$y = 0$ is a horizontal Asymptote.



Question

#Q. $\ln f(x) = \ln(\ln x)$ where $x > e$. Prove that

$$\frac{1}{(m+1)\cdot\ln(m+1)} < f(m+1) - f(m) < \frac{1}{m\cdot\ln(m)} \text{ for } m > e.$$

TAH3


THANK
YOU